

# Lecture 27

Time-Space Lower-bound for *SAT*

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**Note:** Don't confuse  $\mathbf{TISP}(T(n), S(n))$  with  $\mathbf{DTIME}(T(n)) \cap \mathbf{DSPACE}(S(n))$ .

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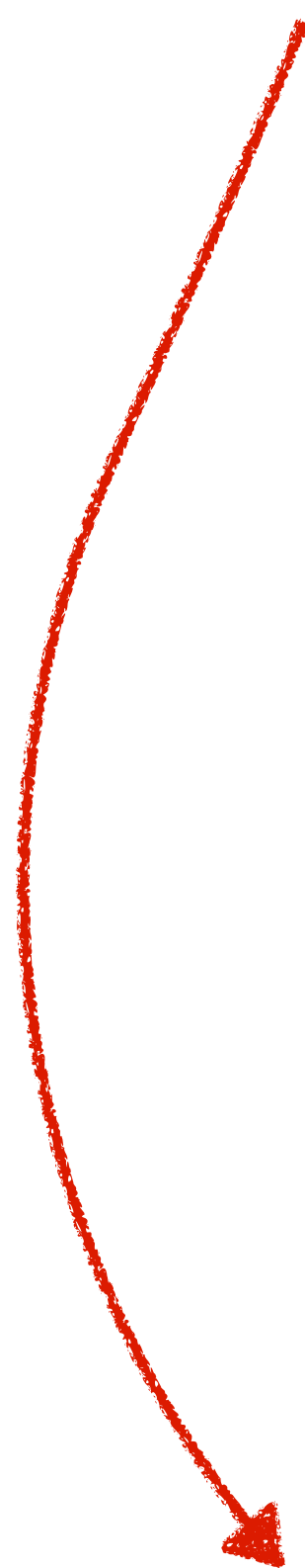
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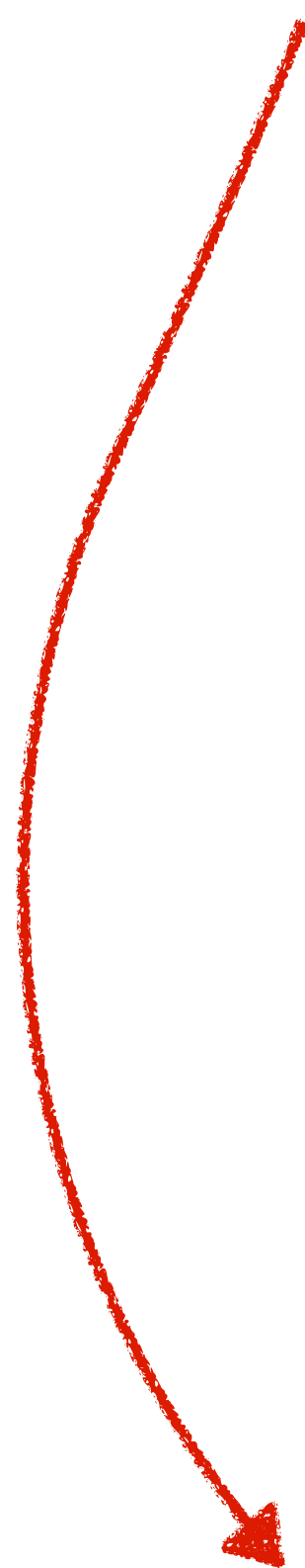
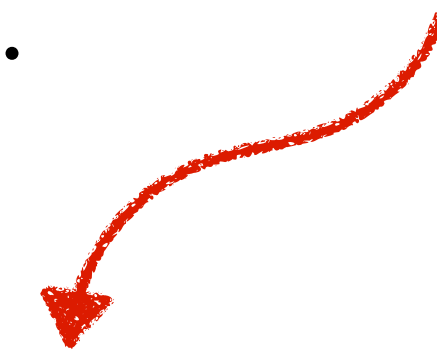
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$\Sigma_2\text{TIME}(n^8)$  algorithm for  $L$  on input  $x$ :

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